J. Fluid Mech. (1992), *vol.* 240, *pp.* 405–432
Printed in Great Britain

Dynamics of singular vortices on a beta-plane

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(Received 28 February 1991 and in revised form 12 December 1991)

A new singular-vortex theory is presented for geostrophic, beta-plane dynamics. The stream function of each vortex is proportional to the modified Bessel function $K_0(pr)$, where p can be an arbitrary positive constant. If p^{-1} is equal to the Rossby deformation scale R_d, then the vortex is a point vortex; for $p^{-1} \neq R_d$ the relative vorticity of the vortex contains an additional logarithmic singularity. Owing to the β-effect, the redistribution of the background potential vorticity produced by the vortices generates a regular field in addition to the velocity field induced by the vortices themselves. Equations governing the joint evolution of singular vortices and the regular field are derived. A new invariant of the motion is found for this system. If the vortex amplitudes and coordinates are set in a particular way then the regular field is zero, and the vortices form a system moving along latitude circles at a constant speed lying outside the range of the phase velocity of linear Rossby waves. Each of the systems is a discrete two-dimensional Rossby soliton and, vice versa, any distributed Rossby soliton is a superposition of the singular vortices concentrated in the interior region of the soliton. An individual singular vortex is studied for times when Rossby wave radiation can be neglected. Such a vortex produces a complicated spiral-form regular flow which consists of two dipoles with mutually perpendicular axes. The dipoles push the vortex westward and along the meridian (cyclones move northward, and anticyclones move southward). The vortex velocity and trajectory are calculated and applications to oceanic and atmospheric eddies are given.

1. Introduction

In recent years a large number of papers has appeared in which the theory of point vortices applied to geophysical hydrodynamics has been developed. For various reasons this model was found attractive. Real atmospheric and oceanic eddies are rather intensive, and the vorticity in the eddies often exceeds the background vorticity (for example, in typhoons or oceanic rings). In many cases the investigation of the dynamics of the individual point vortices and their interaction is simpler than in the analogous problem for distributed regular eddies. Finally, an arbitrary initial field can be represented in the form of superposition of point vortices, and the evolution of the field can be interpreted as a result of the interaction of vortices.

Compared to ordinary two-dimensional hydrodynamics, geophysical hydrodynamics includes a number of additional physical factors (stratification, shear background flow, β-effect, etc.) which strongly affect the structure, motion and interaction of point vortices. An investigation combining all these effects is a very complicated problem, and therefore up to now they have been studied separately. The influence of stratification is only one to have been studied quite thoroughly, and it was found that stratification results in exponential (instead of algebraic) decay of the velocity field of the point vortex (Obukhov 1949; Flierl 1987; Morikawa 1960). Moreover, baroclinic point vortices can change in a vertical direction (Gryanik 1983*a*, *b*, 1988; Gryanik & Tevs 1989).

The inclusion of the β-effect substantially complicates the vortex dynamics. A more general class of singular vortices appears which also possess exponentially decaying velocity fields, but, in general, can be different from point vortices since the stream function of such a vortex is proportional to the modified Bessel function $K_0(pr)$ where the reciprocal decay rate p^{-1} can be different from the Rossby deformation radius (for more details see §2). Several vortices of this kind with appropriately fitted amplitudes and coordinates may form a system moving along latitude circles at a constant speed (lying outside the range of the phase velocity of linear Rossby waves) and generating no velocity field in addition to the velocity fields induced by the vortices themselves (Gryanik 1986, 1988; Reznik 1986; Flierl 1987). There exists a relationship between such systems and two-dimensional Rossby solitons (Reznik 1986; Flierl 1987).

A much more complicated problem arises when the amplitudes and positions of singular vortexes are chosen arbitrarily. Owing to the β-effect, the redistribution of the background potential vorticity produced by the vortices generates a regular velocity field in addition to the velocity field due to the vortices. This regular field interacts with the vortices, affects the vortex trajectories, undergoes variation, etc. Thus, in contrast to classic two-dimensional hydrodynamics, the description of motion of singular vortices cannot be reduced to solving a system of ordinary differential equations but is a complex discrete-continual problem even for a single singular vortex. The problem of a separate point vortex was considered by Bogomolov (1977, 1979, 1985) who investigated the initial stage of motion of a point vortex in a thin rotating spherical layer and showed that a point cyclone (anticyclone) placed in an initially immovable fluid begins to move northwest (southwest).

The aim of the present paper is to elaborate a theory of singular vortices on a β-plane. In §2 we derive equations governing the joint evolution of singular vortices and a regular background flow. In §3 we examine an invariant of motion for such a system. Stationary systems of vortices (discrete Rossby solitions) are considered in §4. The problem of an individual singular vortex is discussed in §5. The case of an intensive individual singular vortex is considered in §§6 and 7. In §8 we consider the problem of an individual singular vortex of moderate intensity in an ocean with a free surface and in the rigid-lid approximation. In §9 we discuss the results and their geophysical applications.

2. Governing equations

The basic equation of the model under consideration is the well-known equation of conservation of potential vorticity (e.g. see Pedlosky 1982)

$$\frac{\partial}{\partial t}(\nabla^2\psi - R_d^{-2}\psi) + \beta\frac{\partial\psi}{\partial x} + J(\psi, \nabla^2\psi) = 0, \tag{1}$$

which is known in plasma physics as the Hasegawa–Mima equation. Here t is time, x and y are coordinate axes directed to east and north respectively, ψ is the stream function, β is the derivative of the Coriolis parameter with respect to the latitude, ∇^2 is the two-dimensional Laplacian, $J(f, g)$ is the Jacobian and R_d is the Rossby radius. The stream function ψ is a sum of two terms, namely a singular component ψ_s and a regular component ψ_r, i.e. $\psi = \psi_r + \psi_s$. The singular component ψ_s is given by the formulae

$$\psi_s = -\frac{1}{2\pi}\sum_{n=1}^{N} A_n K_0(p_n|\boldsymbol{r}-\boldsymbol{r}_n|), \tag{2a}$$

$$\nabla^2\psi_s - R_d^{-2}\psi_s = \sum_{n=1}^{N} A_n\,\delta(x-x_n(t))\,\delta(y-y_n(t)) - \frac{1}{2\pi}\sum_{n=1}^{N} A_n(p_n^2-R_d^{-2})K_0(p_n|\boldsymbol{r}-\boldsymbol{r}_n|). \tag{2b}$$

Here $\delta(z)$ is the Dirac delta function, and $K_m(z)$, $I_m(z)$, $m = 0, 1, \ldots$ are the modified Bessel functions of order m. Thus, the singular component ψ_s describes N singular vortices moving along the trajectories $\boldsymbol{r} = \boldsymbol{r}_n = (x_n(t), y_n(t))$. The vortex amplitudes A_n and parameters p_n that set space scales of the vortex velocity fields can, in principle, depend on time. Note that singular vortices $(2a)$ are point vortices (with relative vorticity concentrated at a point) only for $p_n = R_d^{-1}$. If $p_n \neq R_d^{-1}$, then the vortex has an exponentially decaying component with a logarithmic singularity in addition to the delta function (see $(2b)$). Substituting $\psi = \psi_r + \psi_s$ into (1) and equating to zero the regular part and the parts proportional to $\delta(x-x_n)\ \delta(y-y_n)$, $\delta'(x-x_n)\ \delta(y-y_n)$, and $\delta(x-x_n)\ \delta'(y-y_n)$ we obtain the equations

$$\dot{A}_n = 0, \tag{3a}$$

$$\dot{x}_n = -\left.\frac{\partial(\psi_r+\psi_s^n)}{\partial y}\right|_{\boldsymbol{r}=\boldsymbol{r}_n}, \quad \dot{y}_n = \left.\frac{\partial(\psi_r+\psi_s^n)}{\partial x}\right|_{\boldsymbol{r}=\boldsymbol{r}_n}, \tag{3b, c}$$

$$\begin{aligned}&\frac{\partial\Omega_r}{\partial t} + \beta\frac{\partial\psi_r}{\partial x} + J(\psi_r, \Omega_r)\\ &\quad + \frac{1}{2\pi}\sum_{n=1}^{N} A_n J\{\nabla^2\psi_r - p_n^2\psi_r + [\beta-(p_n^2-R_d^{-2})\dot{x}_n]y + (p_n^2-R_d^{-2})\dot{y}_n x, K_0^n\}\\ &\quad + \frac{1}{8\pi^2}\sum_{m,n}^{N}(p_m^2-p_n^2)A_m A_n J(K_0^n, K_0^m) = 0.\end{aligned} \tag{3d}$$

Here

$$\Omega_r = \nabla^2\psi_r - R_d^{-2}\psi_r, \quad \psi_s^n = -\frac{1}{2\pi}\sum_{m\neq n} A_m K_0(p_m|\boldsymbol{r}-\boldsymbol{r}_m|), \quad K_0^m = K_0(p_m|\boldsymbol{r}-\boldsymbol{r}_m|),$$

$\delta'(z)$ is the derivative of the delta function, and $\dot{a} = \mathrm{d}a/\mathrm{d}t$.

Each of $(3a)$–$(3d)$ has a clear physical meaning. The potential vorticity

$$\nabla^2\psi - R_d^{-2}\psi + \beta y$$

very close to vortex n is a sum of the regular component $\Omega_r + \Omega_s^n + \beta y$ and the singular component

$$A_n\,\delta(x-x_n)\,\delta(y-y_n) + \frac{A_n}{2\pi}(p^2-R_d^{-2})\ln|\boldsymbol{r}-\boldsymbol{r}_n| \tag{4}$$

(see $(2b)$). Since the potential vorticity is conserved in each fluid particle, from (4) it follows that the vortex amplitudes A_n are constant, or, in other words, $(3a)$ is valid. One more important equation follows from conservation of singular vorticity (4), namely

$$\dot{p}_n = 0. \tag{5}$$

Equation (5) means that the lengthscales of velocity fields induced by vortices ($2a$) do not change in time. Equation (5) can also be obtained in another way. We assume that $\dot{p}_m \neq 0$ and investigate the regular field ψ_{r} in the nearest neighbourhood of vortex m. We can find from ($3d$) that there then arises a singularity proportional to $\ln|\boldsymbol{r}-\boldsymbol{r}_m|$ in the regular vorticity $\nabla^2\psi_{\mathrm{r}}-R_{\mathrm{d}}^{-2}\psi_{\mathrm{r}}$. Obviously, this violates the conservation of potential vorticity. Further, ($3b$, c) mean that the motion of a singular vortex is induced by the other singular vortices and the regular component.

The most complicated equation, ($3d$), describes the evolution of the regular component ψ_{r}. This equation contains singular coefficients, and therefore we have to define what kind of solution is considered. We assume that the function ψ_{r} is infinitely differentiable at $\boldsymbol{r} \neq \boldsymbol{r}_n$ and twice differentiable at $\boldsymbol{r} = \boldsymbol{r}_n$. The regular vorticity Ω_{r} is continuous throughout the plane but $\partial\Omega_{\mathrm{r}}/\partial t$, $\boldsymbol{\nabla}\Omega_{\mathrm{r}}$ have singularities at $\boldsymbol{r} = \boldsymbol{r}_n$ (see below), and the singularities in ($3d$) must mutually cancel.

Zabusky & McWilliams (1982) also studied systems of point vortices in the β-plane which are identical with vortices ($2a$, b) for $p_n = R_{\mathrm{d}}^{-1}$. They did not take into account the regular component ψ_{r} and assumed that the vortex amplitudes A_n must change owing to the β-effect when vortices shift along the meridian. A system of equations for the vortex trajectories was introduced. Kono, Horton & Matsuoka (1989) tried to derive the system directly from (1). The preceding discussion shows that this is impossible, and the equations introduced by Zabusky & McWilliams are heuristic (although very useful for modelling some phenomena on the β-plane).

For $\beta = 0$, $p_n = R_{\mathrm{d}}^{-1}$, $n = 1, \ldots, N$, and $\psi_{\mathrm{r}} = 0$, equation ($3d$) is satisfied identically and (3), (5) reduce to the well-known system of ordinary differential equations describing motion of interacting point vortices (e.g. Gryanik 1983). At the same time, if $\psi_{\mathrm{r}} \neq 0$ at some moment, then ψ_{r} also will subsequently remain non-zero since the enstrophy of a regular component is conserved (see the next section). Thus, for $\beta = 0$, $p_n = R_{\mathrm{d}}^{-1}$ the regular component ψ_{r} interacts with the point vortices but is not generated by them. Obviously this is because ($2a$) are point vortices for $p_n = R_{\mathrm{d}}^{-1}$ and the potential vorticity $\nabla^2\psi_{\mathrm{r}}-R_{\mathrm{d}}^{-2}\psi_{\mathrm{r}}$ of the regular component coincides with the total potential vorticity and is conserved in any fluid element distinct from the point vortices. If $\beta = 0$ and $p_n \neq R_{\mathrm{d}}^{-1}$, then (2a) are not point vortices, and $\nabla^2\psi_{\mathrm{r}}-R_{\mathrm{d}}^{-2}\psi_{\mathrm{r}}$ no longer coincides with the total potential vorticity $\nabla^2\psi-R_{\mathrm{d}}^{-2}\psi$ in the fluid element, but depends on the position of the element relative to the singular vortices. Accordingly, if $n > 1$, then the singular vortices generate a regular component ψ_{r} even for $\beta = 0$.

3. Invariant of motion

We shall consider below the case when all p_m are equal, $p_m = p$, and the last sum in ($3d$) is absent. In this case problem (3), (5) has an invariant which, at the same time, is analogous to the Kirchhoff function of n point vortices and the enstrophy of a regular flow. To obtain this we multiply ($3d$) by $\nabla^2\psi_{\mathrm{r}}-p^2\psi_{\mathrm{r}}$ and integrate throughout the plane R. Using the formula

$$\int_R (\nabla^2F-p^2F)\,K_0(pr)\,\mathrm{d}x\,\mathrm{d}y = -2\pi F(0,0), \tag{6}$$

which is valid for any regular function $F(x, y)$, and ($3b$, c) we obtain the invariant

$$L = S+\frac{1}{4\pi}(p^2-R_{\mathrm{d}}^{-2}) \sum_{m \neq n} A_m A_n K_0(pr_{mn})-\beta\sum_n A_n y_n = \text{const.} \tag{7}$$

Here r_{mn} is the distance between the m and n vortices, and

$$S = \frac{1}{2}\int_R [(\nabla^2\psi_r)^2 + (R_d^{-2}+p^2)(\nabla\psi_r)^2 + p^2 R_d^{-2}\psi_r^2]\,dx\,dy. \tag{8}$$

The positive-definite functional S is a linear combination of enstrophy and energy of the regular field ψ_r. It changes, first, owing to the changes in the distances r_{mn} between the vortices and, secondly, owing to the shift of the vortices along the latitude (respectively, the first and second terms on the right-hand side of (7)).

Let us consider an individual singular vortex and differentiate (7) with respect to time t,

$$\dot{S} = \beta A_1 \dot{y}_1. \tag{9}$$

Let the regular component ψ_r be zero at an initial instant t_0. Then the singular vortex begins to generate the regular component ψ_r, and the derivative $\dot{S}$ is positive at times close to t_0, and consequently $A_1\dot{y}_1$ is also positive. As a result, at the initial stage a cyclonic singular vortex ($A_1 > 0$) must move northward and an anticyclonic singular vortex ($A_1 < 0$) must move southward. This effect was demonstrated by numerical and laboratory experiments with regular monopoles (e.g. McWilliams & Flierl 1979; Firing & Beardsley 1976). The process of generation of the regular component and the calculation of the singular vortex trajectory are described in §5.

The functional S determining the intensity of a regular field depends on coordinates of the singular vortices and does not depend on their velocities. The surface

$$F(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = -\frac{p^2-R_d^{-2}}{4\pi}\sum_{m \neq k} A_k A_m K_0(pr_{km}) + \beta\sum_k A_k y_k = \text{const} \tag{10a}$$

has a characteristic point $M_0 = (\boldsymbol{r}_1^0, \ldots, \boldsymbol{r}_N^0)$ at which

$$\frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial y_n} = 0. \tag{10b}$$

It can readily be shown that the coordinates $\boldsymbol{r}_1^0, \ldots, \boldsymbol{r}_N^0$ satisfy the following algebraic system of equations:

$$\sum_{k \neq q} a_{qk}^0 A_k = \frac{2\pi\beta}{p(p^2-R_d^{-2})}, \tag{11a}$$

$$\sum_{k \neq q} b_{qk}^0 A_k = 0, \tag{11b}$$

where $$a_{qk}^0 = \frac{y_q^0 - y_k^0}{r_{qk}^0} K_0'(pr_{qk}^0), \quad b_{qk}^0 = \frac{x_q^0 - x_k^0}{r_{qk}^0} K_0'(pr_{qk}^0), \quad q = 1, \ldots, N.$$

For given amplitudes A_k, $2N$ equations (11a, b) determine $2N$ coordinates x_n^0, y_n^0. If one multiplies (11a) by A_q and performs a summation over q from 1 to N one obtains

$$\sum_{k \neq q} a_{qk}^0 A_k A_q = \frac{2\pi\beta}{p(p^2-R_d^{-2})}\sum_q A_q. \tag{12}$$

Since $a_{qk}^0 = -a_{kq}^0$, the left-hand side of (12) vanishes and

$$\sum_q A_q = 0. \tag{13}$$

Equation (13) is a solvability condition for system (11a, b) and, obviously, means that (11a) are linearly dependent. Similarly, it can easily be shown that (11b) are also linearly dependent. Thus, system (11a, b) contains $2N-2$ linearly independent equations and $2N$ unknowns. Note, however, that a_{qk}^0, b_{qk}^0 depend on differences between coordinates of the vortices, and the number of unknowns in (11 a, b) can be reduced by introducing new unknown quantities $\bar{x}_n^0 = x_n^0 - x_1^0$. $\bar{y}_n^0 = y_n^0 - y_1^0$. The number of these quantities is equal to $2N-2$; that is it coincides with the number of independent equations (11 a, b).

The simplest solution to (11 a, b) is obtained for $N = 2$. By virtue of (13), $A_1 = -A_2$, and we find from (11 b) that $x_1^0 = x_2^0$. Assume that $y_1^0 > y_2^0$, then $a_{12}^0 = K_0'(\mathrm{pr}_{12}^0)$, where $r_{12}^0 = y_1^0 - y_2^0$, and from (11 a) we obtain an equation for r_{12}^0:

$$A_1 = -A_2 = -\frac{2\pi\beta}{p(p^2 - R_\mathrm{d}^{-2})\,K_0'(pr_{12}^0)}. \tag{14}$$

Thus, the simplest configuration of singular vortices corresponding to the point M_0 is a pair of vortices equal in magnitude and opposite in sign; the line segment connecting the vortices is perpendicular to the x-axis. In a more complex case, $N = 3$, the vortices are at the vertices of an equilateral triangle (see below and Reznik 1986). System (11 a, b) has not yet been solved for $N > 3$. For $N = 2$ the point M_0 is a saddle point, but the type of M_0 for $N > 2$ is also as yet unknown.

For $\beta = 0$, $p = R_\mathrm{d}^{-1}$ the functional L is simplified and the conservation law (7) is written as

$$\frac{1}{2}\int [(\nabla^2\psi_\mathrm{r})^2 + 2R_\mathrm{d}^{-2}(\nabla\psi_\mathrm{r})^2 + R_\mathrm{d}^{-4}\,\psi_\mathrm{r}^2]\,\mathrm{d}x\,\mathrm{d}y = \text{const.} \tag{15}$$

By virtue of (15), the enstrophy and the energy of a regular component ψ_r do not increase in time. The physical reason for this was considered in §2.

4. *Stationary systems of vortices*

Let us consider the simplest solutions to (3), (5) for which the regular component ψ_r is zero. In this case (3 d) is satisfied only if $p_1 = p_2 = \ldots = p_N = p$ and

$$\dot{x}_n = U = \frac{\beta}{p^2 - R_\mathrm{d}^{-2}}, \quad \dot{y}_n = 0; \tag{16}$$

that is, all the vortices move along the x-axis with the same velocity. The vortex coordinates are determinated by (3 b, c) which can be written as

$$\sum_{k \neq q} a_{qk} A_k = \frac{2\pi U}{p}, \quad \sum_{k \neq q} b_{qk} A_k = 0, \tag{17 a, b}$$

where $$a_{qk} = \frac{y_q - y_k}{r_{qk}} K_0'(pr_{qk}), \quad b_{qk} = \frac{x_q - x_k}{r_{qk}} K_0'(pr_{qk}), \quad q = 1, \ldots, N.$$

Obviously, system (17 a, b) is identical with (11 a, b). Since the amplitudes A_k in (17 a, b) do not depend on time, the vortex coordinates do not depend on time either (in the coordinate system moving with constant velocity U).

Thus, the solutions with zero regular component ψ_r are stationary vortex systems moving along the latitude at a constant speed. Each of the systems corresponds to a characteristic point M_0 of surface (10) at which the zero absolute minimum of functional S is attained. The phase velocity U is smaller than $-\beta R_\mathrm{d}^2$ for $p < R_\mathrm{d}^{-1}$, is positive for $p > R_\mathrm{d}^{-1}$, and lies outside the range $(-\beta R_\mathrm{d}^2, 0)$ of the phase velocity of the linear Rossby waves. For $r \to \infty$ the velocity field of the vortex system decays proportionally to $\exp(-pr)$, where p is related to U in the following way (see (16)):

$$p^2 = \frac{\beta}{U} + R_\mathrm{d}^{-2}. \tag{18}$$

We know that the two-dimensional Rossby soliton has the same properties (see e.g. Larichev & Reznik 1976; Flierl *et al.* 1980). Therefore one can say that any stationary system of singular vortices (2*a*) is a discrete Rossby soliton. Condition (13) is analogous to the zero-kinetic-moment condition which is valid for any regular Rossby soliton (e.g. Stern 1975; Larichev & Reznik 1976; Flierl, Stern & Whitehead 1983). Conversely, any regular Rossby soliton is a superposition of singular vortices (2*a*) which are concentrated in the finite interior region $\mathscr{D}$ of the solition. To show this we use the fact that the stream function of a Rossby soliton satisfies the equation (e.g. Flierl *et al.* 1980)

$$\nabla^2\psi - p^2\psi = F(\psi + Uy) = \begin{Bmatrix} \hat{F}(\boldsymbol{r}) \neq 0, & \boldsymbol{r} \in \mathscr{D} \\ 0, & \boldsymbol{r} \notin \mathscr{D}, \end{Bmatrix} \tag{19}$$

hence $$\psi = -\frac{1}{2\pi}\int_{\mathscr{D}} \hat{F}(\boldsymbol{r}) K_0(p|\boldsymbol{r}-\boldsymbol{r}'|)\,\mathrm{d}\boldsymbol{r}'.$$

Thus, there exists a direct relationship between singular vortices (2*a*) and two-dimensional Rossby solitons.

The simplest two-dimensional Rossby soliton is the pair oppositely signed vortices that was discussed in the previous section. The solution was obtained and investigated in detail by Gryanik (1986), Reznik (1986), and Flierl (1987). A more complicated three-vortex discrete Rossby soliton was investigated by Reznik (1986). The vortices of such a system form an equilateral triangle; the amplitudes of the vortices depend on the orientation of the triangle and can be written as

$$A_1 = -A_0\cos(\alpha_{13} - \tfrac{1}{3}\pi), \quad A_2 = A_0 \cos\alpha_{13}, \quad A_3 = -A_0\cos(\alpha_{13} + \tfrac{1}{3}\pi). \tag{20}$$

Here $A_0 = 4\pi U/[\surd 3pK_1(pd)]$, d is the length of the side of the triangle, α_{13} is the angle between the x-axis and the line segment connecting vortices 1 and 3. By analogy with the case $N = 2$ (see Gryanik 1986; Reznik 1986; Flierl 1987), one can readily verify that the amplitude A_0 must exceed some threshold value depending on the velocity U. Reznik (1986) and Klyatskin & Reznik (1989) obtained analogous vortex systems in a rotating paraboloid and on a sphere. Note that these stationary systems of vortices (2*a*) exist also for $\beta = 0$, $R_\mathrm{d} = \infty$; that is, in the usual two-dimensional hydrodynamics. Such systems are immovable, and the vortex coordinates are determined by (15) with $U = 0$.

5. An individual singular vortex

System (3), (5) is very complicated, and a complete analysis of it is very difficult. However, we managed to obtain an approximate solution describing the evolution of an individual vortex for finite times. In the case $N = 1$, equations (3*b*–*d*) are written

$$\dot{x}_0 = U = -\left.\frac{\partial\psi_\mathrm{r}}{\partial y}\right|_{\boldsymbol{r}=\boldsymbol{r}_0}, \quad \dot{y}_0 = V = \left.\frac{\partial\psi_\mathrm{r}}{\partial x}\right|_{\boldsymbol{r}=\boldsymbol{r}_0}, \tag{21\,a, b}$$

$$\frac{\partial\Omega_\mathrm{r}}{\partial t} - U\frac{\partial\Omega_\mathrm{r}}{\partial x} - V\frac{\partial\Omega_\mathrm{r}}{\partial y} + \beta\frac{\partial\psi_\mathrm{r}}{\partial x} + J(\psi_\mathrm{r}, \Omega_\mathrm{r}) + \frac{A}{2\pi}J(\nabla^2\psi_\mathrm{r} - p^2\psi_\mathrm{r}, K_0)$$
$$+\frac{A}{2\pi}J\{[\beta - (p^2 - R_\mathrm{d}^{-2})\,U]y + (p^2 - R_\mathrm{d}^{-2})\,Vx, K_0\} = 0. \tag{21\,c}$$

Here A is the vortex amplitude, $\boldsymbol{r}_0(t) = (x_0(t),\ y_0(t))$ is the vortex radius vector,

$K_0 = K_0(pr)$, U and V are the zonal and meridional components of the vortex velocity, and a coordinate system attached to the vortex is used. Problem (21 *a–c*) is solved for the zero initial condition

$$\psi_r|_{t=0} = 0. \tag{21d}$$

There are two timescales in the problem, namely the scale $T_1 = 2\pi p^{-2}/A$ characterizing the singular vortex intensity and the 'wave' scale $T_2 = (\beta/p)^{-1}$ equal to the characteristic time required for a perturbation with lengthscale p^{-1} to travel a distance of the order of p^{-1} due to the β-effect. As will be seen, the motion regime strongly depends on the relationship between the scales T_1 and T_2.

We introduce the non-dimensional variables

$$t^* = \frac{t}{T_1}, \quad \psi_r^* = \frac{\psi_r}{\beta p^{-3}}, \quad (x^*, y^*) = (x, y)p, \quad (U^*, V^*) = \frac{(U, V)}{\beta p^{-2}}$$

and write (21 *a–d*) in non-dimensional form. Equation (3*d*) is written as (the asterisks are dropped)

$$\frac{\partial \Omega_r}{\partial t} + J(\Omega_r, K_0) - \frac{\partial K_0}{\partial x} - (1-\bar{a}^2)\, J(\psi_r + Uy - Vx, K_0) + \alpha\left[\frac{\partial \psi_r}{\partial x} + J(\psi_r + Uy - Vx, \Omega_r)\right] = 0 \tag{22}$$

and the other equations remain unchanged. Here

$$\bar{a} = (pR_d)^{-1}, \ \alpha = T_1/T_2 = 2\pi\beta p^{-3}/A, \ \Omega_r = \nabla^2 \psi_r - \bar{a}^2 \psi_r,$$

and $K_0 = K_0(r)$. The choice of the scales corresponds to the assumption that the first, sixth, and seventh terms in (21*c*) are of the same order and are no smaller than the other terms in (21*c*). This relationship always holds in the nearest neighbourhood of the singular vortex where ψ_s is of the order of $\ln r$ and is much greater than the regular component ψ_r.

We shall distinguish between the following three cases:

a high-intensity vortex when $\alpha \ll 1$, (23*a*)

a moderate-intensity vortex when $\alpha \approx 1$, (23*b*)

and a small-intensity vortex when $\alpha \gg 1$. (23*c*)

6. A high-intensity vortex

We now consider an intensive vortex with a sufficiently large amplitude that (23*a*) is satisfied. Then discarding the terms in square brackets, in (22) we obtain an approximate equation:

$$\frac{\partial \Omega_r}{\partial t} + J(\Omega_r, K_0) - \frac{\partial K_0}{\partial x} - (1-\bar{a}^2)\, J(\Omega_r + Uy - Vx, K_0) = 0. \tag{24a}$$

Unfortunately, (24*a*) is too complicated, and we only managed to study a singular vortex with $1-\bar{a}^2 = O(\alpha) \ll 1$. Such a vortex is close to a point vortex and coincides with the latter when $\bar{a} = 1$. In this case (24*a*) is greatly simplified and can be written as

$$\frac{\partial(\Omega_r + y)}{\partial t} - \frac{1}{r}\frac{\partial K_0}{\partial r}\frac{\partial(\Omega_r + y)}{\partial \theta} = 0. \tag{24b}$$

Here r, θ are polar coordinates centred at the vortex. The solution to (24*b*) has the form

$$\Omega_r + r\sin\theta = Q\left[\theta - \frac{K_1(r)}{r}t, r\right], \tag{25}$$

where $Q(u, r)$ is an arbitrary differentiable function. From initial condition (21 d) and (25) it follows that $Q(\theta, r) = r\sin\theta$, whence we obtain an equation for ψ_r:

$$\nabla^2\psi_r - \bar{a}^2\psi_r = r(\cos bt - 1)\sin\theta - r\sin bt\cos\theta, \tag{26}$$

where $b = b(r) = K_1(r)/r$. The regular solution to (26) satisfying initial condition (21 d) has the form (to within small values of the order of α)

$$\psi_r = \phi_1(r,t)\sin\theta + \phi_2(r,t)\cos\theta, \tag{27a}$$

$$\phi_1 = I_1(r)\int_r^\infty r^2(1-\cos bt)K_1(r)\,\mathrm{d}r + K_1(r)\int_0^r r^2(1-\cos bt)I_1(r)\,\mathrm{d}r, \tag{27b}$$

$$\phi_2 = I_1(r)\int_r^\infty r^2K_1(r)\sin bt\mathrm{d}r + K_1(r)\int_0^r r^2I_1(r)\sin bt\,\mathrm{d}r. \tag{27c}$$

Using (21 a, b) we can easily show that the expression in square brackets in (22) is of O (1) throughout the plane for t of O (1) including the point $r = 0$ despite the singularity of $\nabla\Omega_r$ at $r = 0$. Thus, solution (27) satisfies (22) on the whole plane to within small values of $O(\alpha)$ for t of $O(1)$. Generally, this does not hold for t of O (α^{-1}) since $\partial\Omega_r/\partial r$ is of $O(t)$ and therefore the terms in (22) that are proportional to α may be comparable with the other terms in (22). Knowing the field ψ_r we can easily obtain from (21 a, b) the vortex velocity:

$$U = -\tfrac{1}{2}\beta R_d^2\int_0^\infty r^2\left(1-\cos\frac{\tilde{A}}{R_d^2}bt\right)K_1(r)\,\mathrm{d}r, \tag{28a}$$

$$V = \tfrac{1}{2}\beta R_d^2\int_0^\infty r^2K_1(r)\sin\frac{\tilde{A}}{R_d^2}bt\,\mathrm{d}r, \tag{28b}$$

$$\tilde{A} = A/2\pi,$$

where (28 a, b) are written in dimensional form and are valid for amplitude A of arbitrary sign. Expressions (27) and (28) completely determine the approximate solution of problem (21) for $p = R_d^{-1}(1+O(\alpha))$.

We now consider some properties of this solution. First, note that the regular stream function ψ_r decays exponentially for $r\to\infty$ and is continuous together with the first and second derivatives although the coefficient $K_1(r)/r$ in (22) and (24 b) has a singularity at $r = 0$. Thus, the relative vorticity of the regular field $\nabla^2\psi_r$ remains regular, i.e. no additional singularities appear in the vorticity field. Of course, this fact is in full agreement with conservation of potential vorticity in fluid particles.

Solution (27 a) is a sum of two dipole components with mutually perpendicular axes. The components proportional to $\sin\theta$ and $\cos\theta$ produce motion of the point vortex along the latitude circle and the meridian, respectively. The resulting vortex velocity components are given by (28 a, b). It readily follows from (28 a) that the zonal velocity U is always directed westward. The meridional velocity V conserves its sign (see Appendix A) and is directed northward for a cyclone ($A > 0$) and southward for an anticyclone ($A < 0$). This behaviour of the vortex agrees with the results of numerous computer and laboratory experiments for monopole eddies on the β-plane (e.g. Adem 1956; Firing & Beardsley 1976; McWilliams & Flierl 1979). Formulae (28 a, b) make it possible to obtain the vortex trajectory for $t \ll \alpha^{-1}$ (in dimensional form for $t \ll T_2$), when the vortex has moved through a distance less than its own lengthscale. For $t \ll 1$ we obtain from (27 a, b) (see Appendix A) the expressions

$$U \approx -\tfrac{1}{8}\pi\beta|\tilde{A}|\,t, \quad V \approx -\tfrac{1}{4}\beta\tilde{A}t\ln\left(\frac{|\tilde{A}|}{R_d^2}t\right), \tag{29}$$

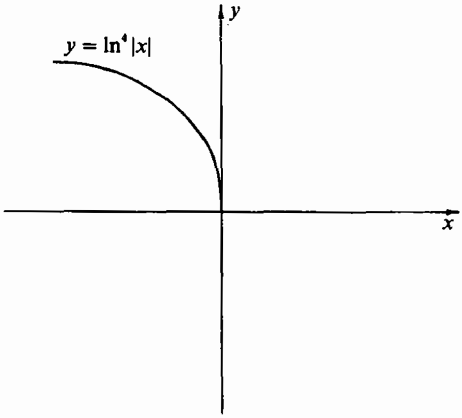

FIGURE 1. Trajectory of an intensive singular vortex with $p = R_d^{-1}(1+O(\alpha))$ on a β-plane (schematically).

whence it follows that at the beginning the vortex moves nearly along the meridian and the vortex velocity increases together with increasing amplitude $|A|$. For large $1 \ll t \ll \alpha^{-1}$ (in dimensional form for $T_1 \ll t \ll T_2$) the velocities can be written in the form

$$U = -\beta R_d^2 + \beta R_d^4\, o\left(\frac{\ln^3[(|\tilde{A}|/R_d^2)\,t]}{|\tilde{A}|t}\right), \tag{30a}$$

$$V = \tfrac{1}{2}\beta R_d^4 \frac{\ln^3[(|\tilde{A}|/R_d^2)\,t]}{\tilde{A}t} + \beta R_d^4\, o\left(\frac{\ln^3[(|\tilde{A}|/R_d^2)\,t]}{|\tilde{A}|t}\right). \tag{30b}$$

It is seen that with increasing t the zonal velocity U no longer depends on the amplitude A and tends to $-\beta R_d^2$ (the limiting phase velocity of Rossby waves) and the meridional velocity V tends to zero. The rate of approach to the limits increases with increasing intensity $|A|$. Accordingly, for $t \gg 1$ the displacements of the vortex along the x- and y-axes are

$$X = -\beta R_d^2 t + o\left(\frac{\beta R_d^4 \ln^4[(|\tilde{A}|/R_d^2)\,t]}{|\tilde{A}|}\right), \tag{31a}$$

$$Y = \frac{\beta R_d^4}{8\tilde{A}} \ln^4 \frac{|\tilde{A}|}{R_d^2} t + o\left(\frac{\beta R_d^4 \ln^4[(|\tilde{A}|/R_d^2)\,t]}{|\tilde{A}|}\right). \tag{31b}$$

It follows from (31) that for large t the vortex trajectory is expressed by the approximate formula

$$Y = \frac{\beta R_d^4}{8\tilde{A}} \ln^4\left(\frac{|\tilde{A}|}{\beta R_d^4}\right)|X|. \tag{32}$$

The trajectory (corresponding to a cyclone) is represented schematically in figure 1.

7. The evolution of material lines

Equation (22) can be written as

$$\left(\frac{\partial}{\partial t} - \alpha U \frac{\partial}{\partial x} - \alpha V \frac{\partial}{\partial y}\right)\Omega + J(-K_0 + \alpha\psi_r, \Omega) = 0, \tag{33}$$

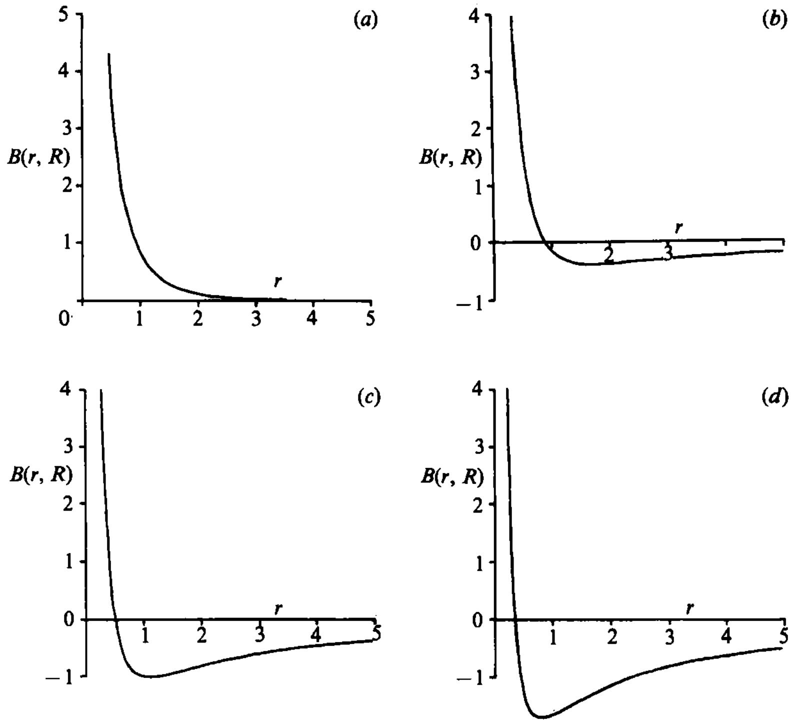

FIGURE 2. Function $B(r,R)$ for different R and $q=2$. (a) $R=0$; (b) $R=-1$; (c) $R=-R_0=-1.83109$; (d) $R=-2.5$.

where the potential vorticity Ω and the vortex displacement along the meridian $y_0(t)$ are

$$\Omega = \Omega_r - \frac{1-\bar{a}^2}{\alpha} K_0(r) + y + \alpha y_0(t), \tag{34}$$

$$y_0(t) = \int_0^t V \, dt. \tag{35}$$

We now consider the evolution of the potential vorticity isolines

$$\Omega(\boldsymbol{r}, t) = \Omega_r - \frac{1-\bar{a}^2}{\alpha} K_0(r) + y + \alpha y_0(t) = \text{const}, \tag{36}$$

coinciding here with material lines. For $t=0$ relation (36) takes the form

$$\Omega(\boldsymbol{r}, 0) = -qK_0(r) + y = R = \text{const}, \tag{37}$$

where the constant R varies from $-\infty$ to ∞, and $q = 1-\bar{a}^2)/\alpha$. Assume that $q>0$ and write (37) in polar coordinates:

$$\sin\theta = \frac{qK_0(r)+R}{r} = B(r,R). \tag{38}$$

For a given R the solution to (38) has two branches:

$$\theta^- = \arcsin B(r,R), \quad \theta^+ = \pi - \arcsin B(r,R), \tag{39}$$

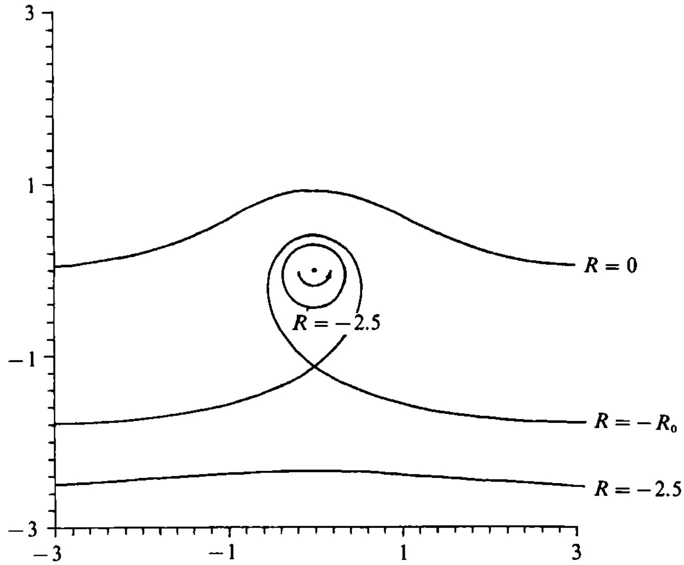

FIGURE 3. Typical vorticity isolines (36) at the initial time $t = 0$ when Ω_r and $y_0(t)$ are equal to zero; $q = 2$, $R_0 = 1.83109$.

where $-\frac{1}{2}\pi \leqslant \arcsin B(r,R) \leqslant \frac{1}{2}\pi$. Solution (39) exists only if $|B(r,R)| \leqslant 1$, i.e. for $r > r_1$ where $r_1 = r_1(R)$ satisfies the equation $B(r_1,R) = 1$,

$$qK_0(r_1)+R = r_1 \tag{40a}$$

(see figure 2). This means that the isoline R is always outside the circle $r = r_1(R)$. The branches θ^-, θ^+ join the point $M_1(r_1,\theta_1), \theta_1 = \frac{1}{2}\pi$ which is the point of tangency of the isoline with the circle $r = r_1(R)$.

The specific value of R is equal to $-R_0$, where R_0 together with the corresponding r^0_{-1} (see figure 2c) satisfies the equations $B(r^0_{-1}, -R_0) = -1$, $B'_r(r^0_{-1}, -R_0) = 0$, i.e.

$$K_1(r^0_{-1}) = \frac{1}{q}, \tag{40b}$$

$$qK_0(r^0_{-1})+r^0_{-1} = R_0. \tag{40c}$$

For $R = -R_0$ the branches θ^- and θ^+ also join at the point $M^0_{-1}(r^0_{-1},\theta^0_{-1})$, $\theta^0_{-1} = -\frac{1}{2}\pi$ and form the separating line L_d (see figure 3) consisting of a closed loop and an unclosed part which approaches the straight line $y = -R_0$ for $r\rightarrow\infty$. The gradient $\nabla\Omega$ vanishes at the saddle point M^0_{-1}.

The isoline with $R > -R_0$ are unclosed and lie above the separating isoline L_d in figure 3. There are two isolines for each $R < -R_0$. The first is closed and lies inside the loop, and the other lies below the separating line in figure 3, and is unclosed. Any unclosed isoline R approaches the line $y = R$ for $r \rightarrow \infty$.

Thus, for $q \neq 0$ there exists near the singular vortex a capture region moving together with the vortex. It follows from (40b) that parameter $r^0_{-1}\rightarrow 0$ as $q\rightarrow 0$ (the singular vortex goes into a point vortex), i.e. the size of the capture region tends to zero. In the case of a point vortex ($q = 0$) the capture region vanishes and the vorticity isolines are straight lines at the initial time $t = 0$.

We now consider the time evolution of the potential vorticity isolines for the singular vortex discussed in the previous section. Since $1-\bar{a}^2 = O(\alpha)$, we have

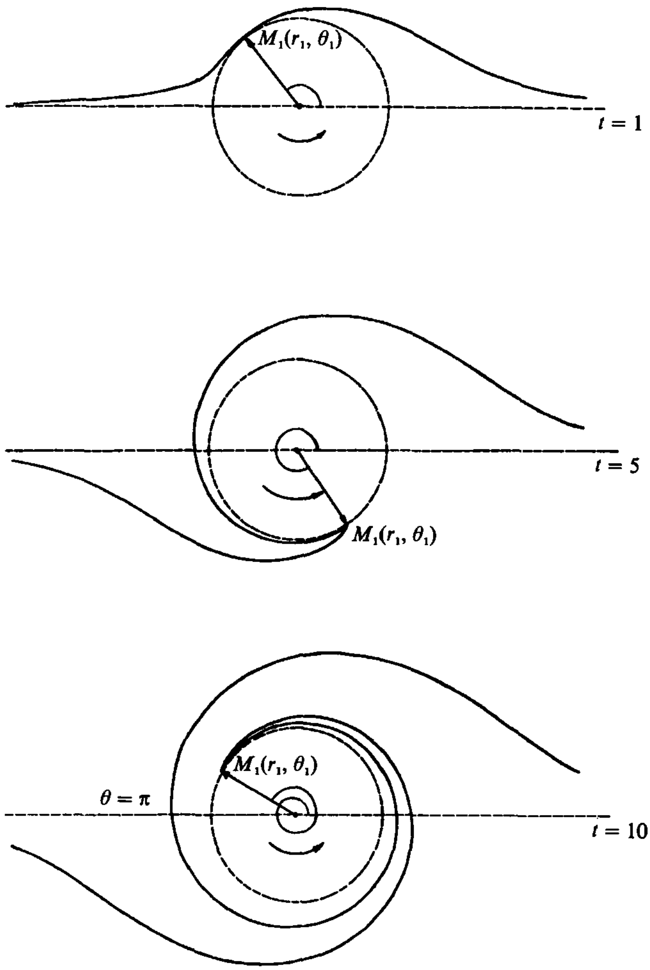

FIGURE 4. Potential vorticity isoline $R = 0$ for $q = 2$ and different times deformed by singular cyclone; here $\theta_1 = b(r_1)t + \frac{1}{2}\pi$ and r_1 satisfies (40a).

$q = (1-\bar{a}^2)/\alpha = O(1)$. By virtue of (26) and (36), an approximate equation of an isoline for $t > 0$ can be written in the form (in the reference frame attached to the vortex)

$$\sin(\theta - bt) = \frac{qK_0(r) + R}{r} = B(r, R). \tag{41}$$

Equation (41) is similar to (38) and the solution to (41) also has two branches for a given R:

$$\theta^+ = bt + \arcsin B(r, R), \quad \theta^- = bt + \pi - \arcsin B(r, R). \tag{42}$$

The common point of these branches $M_1(r_1, \theta_1)$, $\theta_1 = b(r_1)t + \frac{1}{2}\pi$, rotates counterclockwise around the circle $r = r_1(R)$ with constant angular velocity $b(r_1)$. As before, the isoline R is outside the circle $r = r_1(R)$.

As is seen from (42), the form of the isoline is determined by time t and the parameter R, equal to the distance between the vortex and the unperturbed vorticity isoline for unclosed isolines. We now examine how the shape of the isoline depends on the parameter R for a given t.

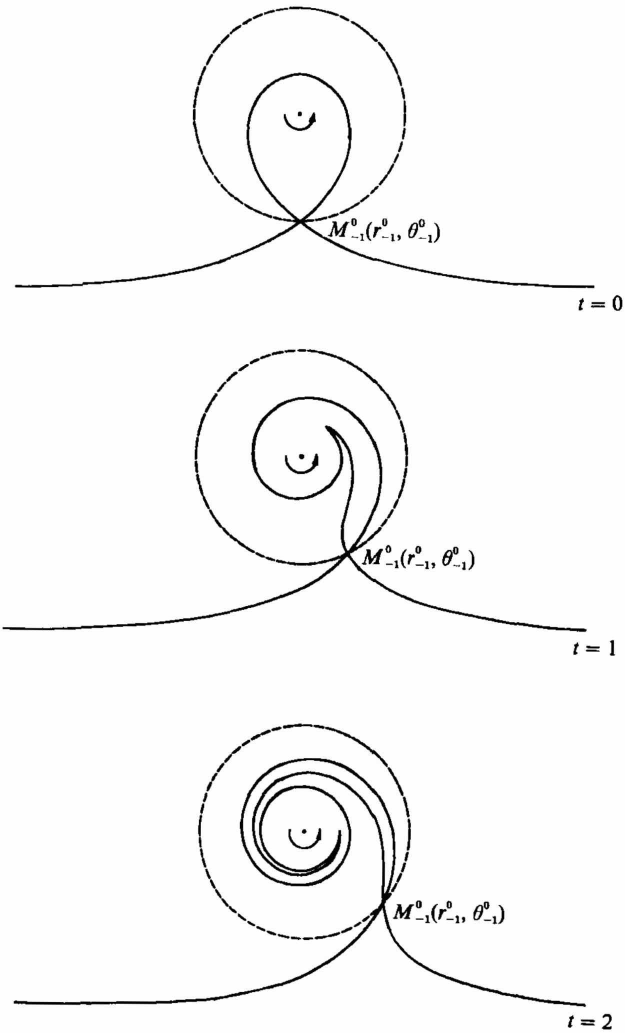

FIGURE 5. Separating isoline for $q = 2$ and different times; $R = -1.83109$.

Let R be non-negative. The derivative

$$\frac{d\theta^-}{dr} = t\frac{db}{dr} - \frac{1}{[1-B^2(r,R)]^{\frac{1}{2}}}\frac{\partial B}{\partial r} \tag{43}$$

is positive for sufficiently small t since $b_r < 0$, $B_r < 0$, and for $r \to \infty$ the functions $b(r)$, $B(r,R)$ decay exponentially and proportionally to r^{-1}, respectively. Thus, $\theta^-(r)$ decreases with decreasing r. For the second branch we have

$$\frac{d\theta^+}{dr} = t\frac{db}{dr} + \frac{1}{[1-B^2(r,R)]^{\frac{1}{2}}}\frac{\partial B}{\partial r}, \tag{44}$$

whence $\theta^+_r < 0$ for any r belonging to the interval $[r_1(R), \infty]$. Accordingly, θ^+ decreases monotonically with increasing r. A potential vorticity isoline for $R \geq 0$ and different t is represented in figure 4. With increasing t the first term on the right-hand side of (43) increases in absolute value and, for sufficiently large t, exceeds the second

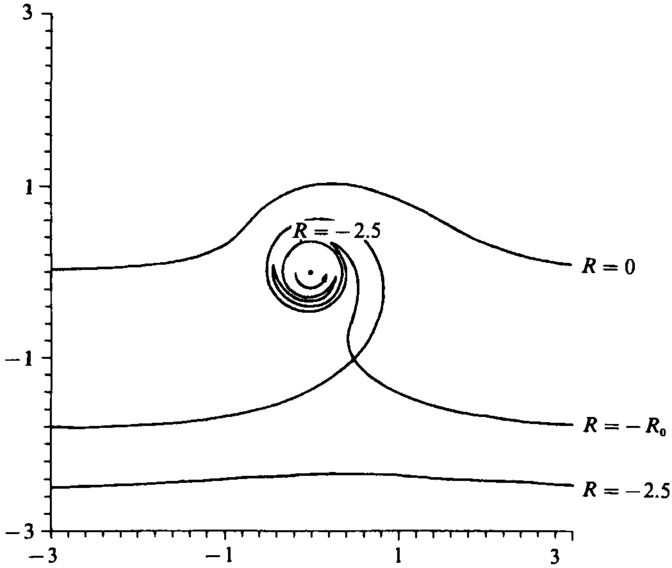

FIGURE 6. Typical vorticity isolines for $q = 2$ and $t = 1$.

term for any r belonging to the interval $[\bar{r}_1, \bar{r}_2]$, where $\bar{r}_1, \bar{r}_2$ are the roots of the equation $\theta_r^- = 0$. In this interval $\theta_r^- < 0$, and θ^- increases with decreasing r. For increasing t the interval $[\bar{r}_1, \bar{r}_2]$ and the angle θ_1 increase and tend to infinity as $t \to \infty$. This means that the vorticity lines with a fixed $R > 0$ must be wound round the vortex more and more strongly with increasing time t (e.g. see figure 4). If we choose the length of the radius vector of the point $\theta^+ = \pi$ (see figure 4) as a characteristic dimension of the spiral line, then by virtue of (41), this parameter is $O(\ln t)$ for large t.

The vorticity isolines with $-R_0 < R < 0$ change in time as in the case $R > 0$. A more interesting process is the time evolution of the separating isoline corresponding to $R = -R_0$. The saddle point M_{-1}^0 has the coordinates $r = r_{-1}^0$ (see (40*b*, *c*) and $\theta = \theta_{-1}^0 = b(r_{-1}^0)t - \frac{1}{2}\pi$, and rotates counterclockwise around the circle $r = r_{-1}^0$ with constant angular velocity $b(r_{-1}^0)$. It readily follows from (40) that $r_{-1}^0 > r_1(-R_0)$, and therefore $b(r_1) > b(r_{-1}^0)$ and the saddle point M_{-1}^0 rotate more slowly than the point M_1. An analysis of the derivatives θ_r^+ and θ_r^- shows that in the interval $[r_1, r_{-1}^0]$ the derivative θ_r^+ is negative, and for sufficiently large t there is a point $r = r^-(t)$ for which $\theta_r^- = 0$, and $\theta_r^- \geqslant 0$ ($\theta_r^- \leqslant 0$) for $r_1 \leqslant r \leqslant r^-(t)$ ($r^-(t) \leqslant r \leqslant r_{-1}^0$). In the region $r > r_{-1}^0$ we have $\theta_r^- < 0$, and there exists a point $r = r^+(t)$ for which $\theta_r^+ = 0$, and $\theta_r^+ \leqslant 0$ ($\theta_r^+ \geqslant 0$) for $r_{-1}^0 \leqslant r \leqslant r^+(t)$ ($r \geqslant r^+(t)$). The separating isoline is represented in figure 5 for different times. As is seen, for sufficiently large time the capture region has a rather complicated form and consists of the circular region $r \leqslant r_1$ and a spiral 'tail', the number of revolutions of the 'tail' round the vortex increasing with time. This evolution is due to the difference in the angular velocities of the points M_1 and M_{-1}^0. For $R < -R_0$ the closed and unclosed isolines change in time like the capture region and the unclosed part of the separating line, respectively. The general pattern of typical vorticity isolines for moderate times is shown in figure 6.

Typical vorticity isolines with different R for a point vortex (when $\bar{a}^2 = 1$, i.e. $q = 0$) are represented in figure 7. The main distinction from the previous case is that all the isolines are unclosed (the capture region is absent) and the number of revolutions of the isoline R around the vortex increases with decreasing $|R|$. The

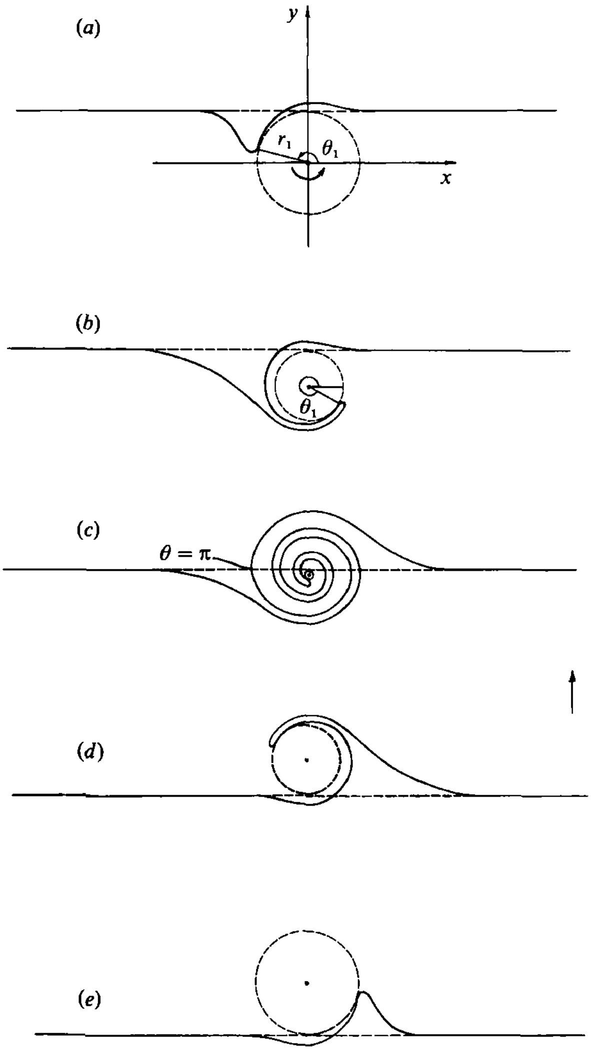

FIGURE 7. Potential vorticity isoline $\Omega + y = R$ deformed by point cyclone (schematically); here $q = 0$, $\theta_1 = b(r_1)t + \frac{1}{2}\pi$, $r_1 = |R|$. (*a*) The vortex is at a large distance $|R|$ to the south of the isoline; (*b*) the vortex is at a moderate distance $|R|$ to the south of the isoline; (*c*) the vortex is at a very small distance $|R|$ from the isoline; (*d*) the vortex is at moderate distance $|R|$ to the north of the isoline; (*e*) the vortex is at a large distance $|R|$ to the north of the isoline. The arrow indicates the direction of motion of the vortex along the meridian.

vorticity line which is at zero distance $R = 0$ from the vortex makes an infinite number of revolutions around the vortex. The time evolution of a fixed-vorticity line R does not differ qualitatively from the evolution in figure 4.

The neglect of terms of the order of α in (36) means that we disregard the effect produced on the material lines by the small meridional displacement $\alpha y_0(t)$ of the vortex and the correction $\bar{\Omega}_r^{(1)}$ of the order of α to the regular vorticity. Note that the terms $\alpha y_0(t)$ and $\bar{\Omega}_r^{(1)}$ must be taken into account in (34) simultaneously although $\bar{\Omega}_r^{(1)}$ decays exponentially for $r \to \infty$. For example, if we include $\alpha y_0(t)$ but neglect $\bar{\Omega}_r^{(1)}$ in (34), then the continuity condition is violated because the parameter R in (41) is replaced by the difference $R - \alpha y_0(t)$, and for $q \neq 0$ separating isoline $R = -R_0$

becomes unclosed for $t > 0$. Similarly, in this case a point vortex ($q = 0$) becomes able to intersect the material lines.

8. A moderate-intensity vortex

We now consider problem (21a,b,d), (22) for a singular vortex of moderate intensity when the parameter α in (22) is of $O(1)$ and p is arbitrary. This is a very difficult problem and we only managed to study the initial stage of the motion using the matched asymptotic expansions method. To find the form of the asymptotic representation we consider solution (27) for $t \to 0$. In the nearest neighbourhood of the vortex of width of $O(t^{\frac{1}{2}})$ (region I) solution (27) has the following approximate form: $\psi_r \approx t^{\frac{3}{2}} \ln t\phi_1(r/t^{\frac{1}{2}}, \theta)$, where $\phi_1(\eta, \theta)$ is a smooth function. In region II lying outside region I, solution (27) is smooth and can be written in another approximate form:

$$\psi_r \approx t\phi_2(r, \theta).$$

The solution for the case of a singular vortex of moderate intensity has a similar structure for $t \to 0$. In region I where the solution changes rapidly in space and time and depends on the variables $\eta = r/t^{\frac{1}{2}}$, θ, t, the asymptotic expansion of the solution has the form

$$\psi_r = \sum_{m=3}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[(m-1)/2]^+} \ln^k t\, \hat{\psi}_{mk}(\eta, \theta), \tag{45}$$

where $[a]^+$ is the integral part of the number a for $a > 0$, and $[a]^+ = 0$ for $a \leqslant 0$. The smooth solution in region II is written as

$$\psi_r = \sum_{m=2}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[(m-2)/2]^+} \ln^k t\, \psi_{mk}(r, \theta). \tag{46}$$

Expansions (45), (46) are self-consistent in the sense that their substitution into (22) yields no terms distinct from those in (45), (46) (see Apprendix B). The peculiarities of the matching procedure for (45), (46) are described in Appendix C; here we discuss the lowest-order solution.

Assuming that $\alpha = 1$, substituting (45) and (46) into (22), and collecting the terms with the same m, k we find, after some transformations, that

$$\tilde{\nabla}^2 \hat{\psi}_{31} = \eta s_{31}\left(\theta - \frac{1}{\eta^2}\right), \tag{47a}$$

$$\tilde{\nabla}^2 \hat{\psi}_{30} = \eta\left[s_{30}\left(\theta - \frac{1}{\eta^2}\right) - \sin\theta\right] + 2\eta \ln \eta s_{31}\left(\theta - \frac{1}{\eta^2}\right), \tag{47b}$$

$$\nabla^2 \psi_{20} - \bar{a}^2 \psi_{20} = -K_1(r)\cos\theta. \tag{47c}$$

where s_{30}, s_{31} are arbitrary functions and

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta} - \frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}.$$

To find the functions s_{30}, s_{31}, $\hat{\psi}_{30}$, $\hat{\psi}_{31}$, ψ_{20} it is necessary to supplement (47a–c) with matching conditions; the detailed discussion of these conditions is given in Appendix C and here we only give the results.

First, we match the expansions for the relative vorticity Ω_r in regions I and II and find that

$$s_{31}\left(\theta-\frac{1}{\eta^2}\right)=0,\quad s_{30}\left(\theta-\frac{1}{\eta^2}\right)=\sin\left(\theta-\frac{1}{\eta^2}\right). \tag{48}$$

The solution to (47*a*–*c*) satisfying the matching conditions for expansions (45), (46) is written as

$$\hat{\psi}_{31}=-\tfrac{1}{4}\eta\cos\theta, \tag{49a}$$

$$\hat{\psi}_{30}=\eta(\tfrac{1}{8}\pi\sin\theta+\hat{B}_{30}^{(1)}\cos\theta)+\varphi_{30}^{(1)}(\eta)\sin\theta+\varphi_{30}^{(2)}(\eta)\cos\theta, \tag{49b}$$

$$\psi_{20}=\frac{1}{1-\bar{a}^2}[\bar{a}K_1(\bar{a}r)-K_1(r)]\cos\theta, \tag{49c}$$

where

$$\hat{B}_{30}^{(1)}=-\tfrac{1}{4}(3C+\ln\tfrac{1}{4})+\tfrac{1}{4}+\frac{\bar{a}^2\ln\bar{a}}{2(1-\bar{a}^2)}, \tag{50}$$

$$\varphi_{30}^{(i)}=\tfrac{1}{2}\eta\int_0^\eta R_i(z)\,\mathrm{d}z-\frac{1}{2\eta}\int_0^\eta z^2R_i(z)\,\mathrm{d}z,\quad i=1,2, \tag{51}$$

$$R_1(\eta)=\eta\left(\cos\frac{1}{\eta^2}-1\right),\quad R_2(\eta)=-\eta\sin\frac{1}{\eta^2};$$

C is Euler's constant. Using (45), (46), (49*a*, *b*), (50), (51) one can find the zonal and meridional velocities of the singular vortex:

$$U=-\tfrac{1}{8}\pi t+O(t^{\frac{3}{2}}\ln t), \tag{52a}$$

$$V=-\tfrac{1}{4}t\ln t+t\hat{B}_{30}^{(1)}+O(t^{\frac{3}{2}}\ln t). \tag{52b}$$

Equations (49), (52) determine completely the regular field ψ_r up to within $O(t^2\ln t)$ for small t. In principle, the above procedure can be continued to find the asymptotic solution to within an arbitrarily small error.

As in the case of an intensive vortex, solution (49), (52) decays exponentially for $r\to\infty$ and is a sum of two dipole components with mutually perpendicular axes. The intensity of the component proportional $\cos\theta$ exceeds that of the $\sin\theta$ component and, therefore, the meridional vortex velocity is much smaller than the zonal velocity. This means that the vortex always moves nearly along the meridian at the initial stage of motion (compare with (29)). It is interesting that the structure of the dipoles near the vortex (in region I) and consequently the vortex velocity (52) do not depend on the vortex scale p^{-1} in the lowest order. The vortex velocity also coincides with the initial velocity of a logarithmic point vortex on a rotating sphere calculated by Bogomolov (1985). Thus, the initial dynamics of a singular vortex is local in the sense that the structure of the lowest-order regular field in the nearest neighbourhood of the vortex is determined only by the logarithmic singularity in ψ_s and does not depend on the field outside that neighbourhood.

The resulting solution (49), (52) is valid for $\bar{a}\neq 0, 1$, and in the case $\bar{a}=1$ when a singular vortex goes into a point vortex the solution slightly changes, namely the function ψ_{20} in (49*c*) and the coefficient $\hat{B}_{30}^{(1)}$ in (49*b*) are replaced by

$$\psi_{20}=\left[I_1(r)\int_r^\infty zK_1^2(z)\mathrm{d}z+K_1(r)\int_0^r zK_1(z)I_1(z)\mathrm{d}z\right]\cos\theta, \tag{53}$$

$$\hat{B}_{30}^{(1)}=-\tfrac{1}{4}(3C+\ln\tfrac{1}{4}). \tag{54}$$

The solution changes more substantially in the case $\bar{a} = 0$, which corresponds to the rigid-lid approximation. Here the formal difficulty lies in the fact that expansion (46) is inapplicable for large r since the coefficients $\psi_{mk}(r, \theta)$ in (46) decay not exponentially (as in the case $\bar{a} \neq 0$) but algebrically and the energy of the regular component is infinite for arbitrarily small t (for more details see Appendix D). Physically, the inapplicability of (46) relates to the fact that in the rigid-lid approximation the maximum group velocity of Rossby waves becomes infinite and their influence is significant in the far-field region $r \gg 1$ even for small t. Only very long Rossby waves can reach the region $r \gg 1$ for $t \rightarrow 0$, and therefore the solution must change slowly in space for $r \gg 1$ and $t \rightarrow 0$. Since the radiation term $\partial\psi_r/\partial x$ is of the order of $\partial\nabla^2\psi_r/\partial t$ for $r \gg 1$, the solution in the far-field region depends on the stretched coordinates

$$(X, Y) = t(x, y). \tag{55}$$

So, besides the regions I and II (where, as before, expansions (45), (46) are valid), in the case $\bar{a} = 0$ there appears a new region III where $r = O(t^{-1})$ and the solution is written in form of the expansion

$$\psi_r = t^2\bar{\psi}_{20}(X, Y) + t^3\bar{\psi}_{30}(X, Y) + t^4\bar{\psi}_{40}(X, Y) + \dots. \tag{56}$$

The solution in region I remains unchanged, but in region II it changes so that

$$\psi_{20} = \left[\frac{1}{r} - K_1(r)\right] \cos\theta. \tag{57}$$

Substituting (56) into (22) and matching (46) and (56) we obtain

$$\bar{\psi}_{20}(X, Y) = 2\pi \frac{\partial \bar{G}}{\partial X}, \tag{58a}$$

$$\bar{G}(X, Y) = -\frac{1}{\pi}\frac{1}{R^{\frac{1}{2}}}\int_0^\infty \frac{J_1(2[R(u^2+\cos^2(\frac{1}{2}\theta))]^{\frac{1}{2}})}{[(u^2+1)(u^2+\cos^2(\frac{1}{2}\theta))]^{\frac{1}{2}}}\,\mathrm{d}u, \tag{58b}$$

where $R = (X^2 + Y^2)^{\frac{1}{2}}$ (for more detail see Appendix D).

For $r \rightarrow \infty$ we have $\partial G/\partial x = O(r^{-\frac{3}{2}})$ (Kamenkovich 1989), and therefore the energy of the regular component is finite. Note that the resulting solution has a rather complicated structure and cannot be represented as a sum of two dipoles in the far-field region III.

9. Summary and discussion

The preceding analysis shows that the singular vortices on a β-plane substantially differ from the well-known point vortices in two-dimensional hydrodynamics. The redistribution of the background potential vorticity produced by the vortices, generates a regular (i.e. without singularities) field in addition to the velocity field induced by the vortices themselves. The system of equations describing the joint evolution of vortices and the regular field is obtained in §2. An analysis of the system shows that if the vortex amplitudes and coordinates are not set in an appropriate manner, then the regular field is non-zero, and therefore the theory developed by Zabusky & McWilliams (1982) and Kono *et al.* (1989) is heuristic.

In the case when all the singular vortices possess the same spatial structure (i.e. $p_1 = p_2 = \dots = p_N$) the system of equations is substantially simplified and possesses an integral of motion (§3). An analysis of the integral shows that in the case of an individual singular vortex the regular component moves the vortex along the meridional direction (cyclone travel northward, and anticyclone travel southward). The absolute minimum of energy of a regular component (equal to zero) is realized in stationary systems moving along the latitude circle at a constant speed lying outside the range of phase velocities of linear Rossby waves (§4). Each of the systems is a discrete two-dimensional Rossby soliton and, vice versa, any distributed Rossby soliton is a superposition of the singular vortices concentrated in the interior region of the soliton.

An individual singular vortex was studied in §§5–8. The suggested theory is valid for times

$$t \ll (\beta/p)^{-1} \tag{59}$$

when Rossby wave radiation can be neglected. For these times the motion of the singular vortex and the regular field are approximately described by (24*a*). This regime attains a developed stage for a high-intensity vortex when the advective timescale $T_1 = 2\pi/p^2A$ is much smaller than the wave timescale $T_2 = (\beta/p)^{-1}$, i.e. $\alpha = T_1/T_2 \ll 1$. We have examined this case for $p = R_d^{-1}(1+O(\alpha))$ when a singular vortex is closed to a point vortex (§6). At the initial stage, for $t \ll T_1$, a region of rapid space–time variations of width of the order of $t^{\frac{1}{2}}$ appears in the neighbourhood of the vortex, and outside the region the field is smooth. In the course of time the region of rapid variations extends monotonically, and its size increases proportionally to $\ln t$ for $T_1 \ll t \ll T_2$. The regular component is a sum of two dipoles with mutually perpendicular axes. The advective interaction of the distributed vortices in the dipoles generates motion of the point vortex along both the latitude and longitude. With increasing time the intensity of the dipole with axis parallel to latitude tends to a constant value and the intensity of the dipole with meridional axis tends to zero. Accordingly, the point-vortex velocity varies from a nearly meridional small velocity at the initial stage of motion to an almost zonal velocity close to the Rossby wave drift velocity $-\beta R_d^2$ for $t \gg T_1$. In the mid latitudes R_d is approximately equal to 50 km for the ocean and to 2000 km for the atmosphere, and therefore at the latitude 45° the drift velocity is equal to approximately 4 cm/s for the ocean and 65 m/s for the atmosphere. Thus, the vortex velocity induced by the β-effect can be of the order of (in the ocean) or exceed (in the atmosphere) the leading flow velocity. This means that the β-effect must be taken into account in investigations of motion of intensive eddies in the ocean and atmosphere (e.g. of rings and typhoons).

It is important to note that the nearer the velocity of a localized eddy is to the Rossby wave drift velocity $-\beta R_d^2$, the weaker the destruction of the eddy by Rossby wave radiation. This is because an eddy moving with velocity close to $-\beta R_d^2$ radiates long almost non-dispersive Rossby waves with phase and group velocities that are also close to the drift velocity. From the above theory it follows that the more intensive the vortex is, the nearer its final velocity approaches the drift velocity. Thus, the Rossby wave radiation destroys more effectively localized eddies with small amplitudes than those with large amplitudes. This conclusion is confirmed by the results of numerical experiments described by Smith & Reid (1982), Horton (1989), and Kolchik, Reznik & Stepanyants (unpublished).

An analysis of material lines (potential vorticity isolines) given in §7 shows that a singular vortex with $p \neq R_d^{-1}$ captures ambient water and involves in rotational motion a water volume increasing with time. Under the action of the vortex, a material line undergoes spiral-like deformation which increases with increasing time and/or decreasing distance between the vortex and the material line. Note that at the lowest order the vortex trajectory and velocity do not depend on the parameter $q = (1-p^{-2}R_{\mathrm{d}}^{-2})/\alpha$, which is a measure of the deviation of the vortex from a point vortex (§6). At the same time the form and the evolution of material lines depend strongly on this parameter; for example, the capture region vanishes for $q = 0$ (for a point vortex) and increases with increasing q.

The initial evolution of vortices with arbitrary scales p^{-1} and small or moderate amplitudes (when $\alpha = T_1/T_2 \gtrsim 1$) were considered in §8. In this case the regime described by (24*a*) is realized only for $R_{\mathrm{d}}^{-1} \neq 0$ and $t \ll T_1$, then the Rossby radiation and self-interaction of a regular field must be taken into account for $t \gtrsim T_1$. The comparison of solutions for different $\bar{a}$ shows that, to within small magnitudes, at the initial stage the regular field in the nearest neighbourhood of the singular vortex (region I) and the vortex velocity do not depend on $\bar{a}$. Taking account of the terms of the order of t in the V-component, it follows from (52) that a decrease of the vortex size p^{-1} leads to an increase in the vortex meridional velocity. The flow field outside region I depends strongly on $\bar{a}$, and the decay rate of the field for $r \to \infty$ decreases with decreasing $\bar{a}$. This is accounted for by the increase of maximum group velocity of Rossby waves with decreasing $\bar{a}$.

In the rigid-lid approximation ($R_{\mathrm{d}}^{-1} = 0$) the maximum group velocity of Rossby waves is infinite and the Rossby wave radiation cannot be neglected even for $t \to 0$. Accordingly, (24) is inapplicable throughout the plane and the radiation term $\alpha\, \partial\psi_{\mathrm{r}}/\partial x$ in (22) is important in the far-field region where $r = O(t^{-1})$. The radiation field changes slowly in space since only very long Rossby waves can reach the region $r \gg p^{-1}$ for $t \to 0$. The field has a rather complicated structure, and, in contrast to the stream-function field in the region $r \lesssim p^{-1}$, it cannot be represented as a sum of two dipoles.

Note that the nonlinear problem (22), (21*d*) is rather complicated even for $t \to 0$. For some details relating to the solution of this problem see Appendixes B–D. The suggested method can be used for investigating the initial stage of motion of a singular vortex in arbitrary regular flows.

Many questions connected with the above theory arise. Are the discrete Rossby solitons stable? Can the singular vortices combine spontaneously to form discrete Rossby solitons? What is the nature of interaction between vortices with different scales p_n? What is the effect of the Rossby wave radiation on the vortex dynamics? What are the stochastic regimes in systems of such vortices? etc. These remain to be solved.

The author expresses his gratitude to Professors G. K. Korotaev, V. M. Kamenkovich, V. M. Volosov and Drs S. Meacham, S. V. Musylev, D. Y. Lysanov for valuable discussions.

Appendix A

We rewrite (28*b*) in non-dimensional form

$$V = \int_0^\infty r^2 K_1(r) \sin bt \, \mathrm{d}r, \quad b = (1/r) K_1(r), \tag{A 1}$$

and replace the variable r by the variable $b(r)$; it follows from (A 1) that

$$V = \int_0^\infty F(b) \sin bt \, \mathrm{d}b, \tag{A 2}$$

where
$$F(b) = \phi[r(b)] = \frac{r^4(b)}{2+r(b)\Gamma(b)}, \quad \Gamma(b) = \frac{K_0[r(b)]}{K_1[r(b)]}. \tag{A 3}$$

The derivative dF/db is given by
$$\frac{dF}{db} = \frac{d\phi}{dr}\frac{dr}{db} = \frac{dr}{db}\frac{8r^3(b)+2r^4(b)\Gamma(b)+r^4(b)[\Gamma(b)-r(b)\,\Gamma_r'(b)]}{[2+r(b)\,\Gamma(b)]^2}. \tag{A 4}$$

Using the formula $K_0(r) = -K_1(r)$ it can readily be shown that
$$\Gamma - r\Gamma_r' = \frac{r(K_1^2-K_0^2)}{K_1^2}.$$

By virtue of formula 9.6.24 in Abramowitz & Stegun (1964), we have $K_1(r) > K_0(r)$, and hence $\Gamma - r\Gamma_r' > 0$. Thus, $d\phi/dr > 0$ and $dF/db < 0$ since $dr/db < 0$. Hence, the function $F(b)$ decreases monotonically with increasing b. Let us represent integral (A 2) in the form
$$V = \sum_{n=0}^{\infty} \mathscr{T}_n, \quad \mathscr{T}_n = \int_{\pi n}^{\pi(n+1)} F(b)\sin bt\,db. \tag{A 5}$$

Taking into account that the function $F(b)$ is positive and monotonic, one can easily show that (i) $\mathscr{T}_{2m} > 0$, $\mathscr{T}_{2m+1} < 0$; and (ii) $|\mathscr{T}_{2m}| > |\mathscr{T}_{2m+1}|$, whence $V > 0$.

To obtain the limiting cases (29), (30) we replace r with $b(r)$ in integrals (28*a*, *b*) and then use the ordinary methods of approximate evaluation of integrals with oscillating integrands (e.g. Olver 1974).

Appendix B

Consider the expansion
$$S = \sum_{m=3}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[(m-1)/2]^+} \ln^k t\, S_{mk}(\eta,\theta). \tag{B 1}$$

LEMMA I. *The product $S = t^{-\frac{1}{2}p}\ln^q tS$, $p > 0$, $0 \leqslant q \leqslant [\frac{1}{2}p]^+$ can be represented in the form of expansion (B 1).*

One can write the following simple relations:
$$S_1 = \sum_{m=3}^{\infty} t^{(m+p)/2} \sum_{k=0}^{[(m-1)/2]^+} \ln^{k+q} tS_{mk} = \sum_{n=p+3}^{\infty} t^{\frac{1}{2}n} \sum_{k=q}^{[(n-p-1)/2]^+ + q} \ln^k tS'_{nk}, \tag{B 2}$$

where $S'_{nk} = S'_{nk}(\eta,\theta)$. Since $[\alpha-\beta]^+ = [\alpha]^+ - [\beta]^+$ for $\alpha > \beta \geqslant 0$,
$$\left[\frac{n-p-1}{2}\right]^+ + q = \left[\frac{n-1}{2}\right]^+ - \left[\frac{p}{2}\right]^+ + q \leqslant \left[\frac{n-1}{2}\right]^+. \tag{B 3}$$

Lemma I is proved.

LEMMA II. *The product of two expansions of the form of (B 1) can be represented as (B 1).* Consider the product
$$TS = \left(\sum_{m=3}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[(m-1)/2]^+} \ln^k t\, T_{mk}\right)\left(\sum_{n=3}^{\infty} t^{\frac{1}{2}n} \sum_{l=0}^{[(n-1)/2]^+} \ln^l t\, S_{nl}\right).$$

One can readily show that

$$TS = \sum_{m=3}^{\infty} \sum_{n=3}^{\infty} t^{(m+n)/2} \sum_{k=0}^{k_m} \sum_{l=0}^{l_n} \ln^{k+l} t\, T_{mk} S_{nl}$$
$$= \sum_{m,n=3}^{\infty} t^{(m+n)/2} \sum_{r=0}^{k_m+l_n} \ln^r t\, T'_{mnr},$$

where $k_m = [(m-1)/2]^+$, $l_n = [(n-1)/2]^+$. We have

$$k_m + l_n = \left[\frac{m-1}{2}\right]^+ + \left[\frac{n-1}{2}\right]^+ \leqslant \left[\frac{m+n-2}{2}\right]^+,$$

since $[\alpha+\beta]^+ \geqslant [\alpha]^+ + [\beta]^+$ for α, $\beta > 0$. Finally, we obtain

$$TS = \sum_{m,n=3}^{\infty} t^{(m+n)/2} \sum_{r=0}^{[(m+n-2)/2]^+} \ln^r t\, T''_{mnr} = \sum_{q=6}^{\infty} t^{\frac{1}{2}q} \sum_{r=0}^{[(q-2)/2]^+} \ln^r t\, T'''_{qr}.$$

Whence Lemma II follows.

Rewrite (22) in terms of $\eta = r/t^{\frac{1}{2}}$, θ, t, substitute expansion (45) into the resulting equation, and consider expansions of individual terms. Using Lemmas I, II one can readily show that all the terms can be written in the form of (B 1). Thus, the substitution of expansion (45) into (22) generates no terms different from those in (45), i.e. expansion (45) is self-consistent.

To carry out an analogous investigation of expansion (46) we substitute it into (22). The derivative $\partial\Omega_r/\partial t$ can be represented in the following form:

$$\frac{\partial \Omega_r}{\partial t} = \sum_{n=0}^{\infty} t^{\frac{1}{2}n} \sum_{k=0}^{[\frac{1}{2}n]^+} \ln^k t\, T'_{nk}. \tag{B 4}$$

Let us consider the product Π of two expansions of the form of (46):

$$\bar{T} = \sum_{m=2}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[(m-2)/2]^+} \ln^k t\, \bar{T}_{mk},$$
$$\bar{S} = \sum_{n=2}^{\infty} t^{\frac{1}{2}n} \sum_{l=0}^{[(n-2)/2]^+} \ln^l t\, \bar{S}_{nl}. \tag{B 5}$$

The product Π is equal to

$$\Pi = \bar{T}\bar{S} = \sum_{m,n=2}^{\infty} t^{(m+n)/2} \sum_{r=0}^{\bar{k}_m+\bar{l}_n} \ln^r t\, T'_{mnr},$$

where $\bar{k}_m = [(m-2)/2]^+$, $\bar{l}_n = [(n-2)/2]^+$ whence we conclude that

$$\Pi = \sum_{m,n=2}^{\infty} t^{(m+n)/2} \sum_{r=0}^{[(m+n-4)/2]^+} \ln^r t\, T''_{mnr} = \sum_{q=4}^{\infty} t^{\frac{1}{2}q} \sum_{r=0}^{[(q-4)/2]^+} \ln^r t\, T''_{qr}. \tag{B 6}$$

It follows from (45) that

$$(U, V) = \sum_{m=2}^{\infty} t^{\frac{1}{2}m} \sum_{k=0}^{[\frac{1}{2}m]^+} \ln^k t (U_{mk}, V_{mk}), \tag{B 7}$$

whence

$$U\bar{T} = \sum_{q=4}^{\infty} t^{\frac{1}{2}q} \sum_{r=0}^{[(q-2)/2]^+} \ln^r t\, \tilde{T}_{qr}. \tag{B 8}$$

Using (B 6), (B 8) we see that the sum

$$J(\psi_r, \nabla^2\psi_r) + J(Ur\sin\theta - Vr\cos\theta, \Omega_r) + \frac{\partial\psi_r}{\partial x} + K_1\cos\theta$$

$$+\frac{1}{r}K_1\frac{\partial\Omega_r}{\partial\theta} - (1-i^2)\,J(\psi_r + Uy - Vx, K_0)$$

is represented in the form (B 4), i.e. expansion (46) is self-consistent.

Appendix C

The matching procedure is not simple here since different terms in expansions (45) and (46) can be of the same order in the intermediate region where

$$r = \zeta t^{\gamma},\ 0 < \gamma < \tfrac{1}{2},\ \zeta = O(1).$$

However, it turns out that this can take place only for the terms of the same degree m. In the intermediate region the general term of (46) takes the form

$$t^{\frac{1}{2}m}\ln^k t f_{mk}(\zeta t^{\gamma}, \theta),\ 0 < \gamma < \tfrac{1}{2}.$$

Two terms with different m_1, k_1, and m_2, k_2 are of the same order if

$$\frac{t^{\frac{1}{2}m_1}\ln^{k_1} t f_1(\zeta t^{\gamma}, \theta)}{t^{\frac{1}{2}m_2}\ln^{k_2} t f_2(\zeta t^{\gamma}, \theta)} = c(\zeta, \gamma, \theta) + o(1) \tag{C 1}$$

for $t \to 0$. Here $f_i = f_{m_i k_i}(\zeta t^{\gamma}, \theta)$, $i = 1, 2$ and $c(\zeta, \gamma, \theta)$ is of the order of 1. Introducing the variable $u = t^{\gamma}$ we can rewrite (C 1) in the form

$$\varphi(\zeta u, \theta) = u^{(m_2 - m_1)/2\gamma}\ln^{k_2 - k_1} u[c(\zeta, \gamma, \theta) + o(1)]\gamma^{k_1 - k_2}, \tag{C 2}$$

where $\varphi(\zeta u, \theta) = f_1(\zeta u, \theta)/f_2(\zeta u, \theta)$. It readily follows from (C 2) that

$$\ln\varphi(\zeta u, \theta) \approx \frac{m_2 - m_1}{2\alpha}\ln u \tag{C 3}$$

for $u \to 0$. Obviously, (C 3) is possible only for $m_1 = m_2$. The investigation of (45) is similar to the above. Thus, the whole sums

$$t^{\frac{1}{2}m}\sum_{k=0}^{[(m-1)/2]^+}\ln^k t\hat{\psi}_{mk}(\eta, \theta),\quad t^{\frac{1}{2}m}\sum_{k=0}^{[(m-2)/2]^+}\ln^k t\psi_{mk}(r, \theta)$$

must be matched and not just separate terms of expansions (45) and (46). The lowest-order sums are written as

$$t^{\frac{3}{2}}\ln t\hat{\psi}_{31}(\eta, \theta) + t^{\frac{3}{2}}\hat{\psi}_{30}(\eta, \theta) \quad \text{and} \quad t\psi_{20}(r, \theta). \tag{C 4}$$

We first determine the vorticity Ω_r. The corresponding expansions in regions I and II have the form

$$\Omega_r = t^{\frac{1}{2}}\ln t\hat{\Omega}_{31}(\eta, \theta) + t^{\frac{1}{2}}\hat{\Omega}_{30}(\eta, \theta) + \ldots, \tag{C 5a}$$

$$\Omega_r = t\Omega_{20}(r, \theta) + t^{\frac{3}{2}}\Omega_{30}(r, \theta) + \ldots, \tag{C 5b}$$

where $\hat{\Omega}_{31} = \tilde{\nabla}^2\tilde{\psi}_{31}$, $\hat{\Omega}_{30} = \tilde{\nabla}^2\hat{\psi}_{30}$, $\Omega_{20} = \nabla^2\psi_{20} - \bar{a}^2\psi_{20}$, and $\Omega_{30} = \nabla^2\psi_{30} - \bar{a}^2\psi_{30}$. Rewriting (C 5a, b) in terms of the intermediate variable $\zeta = rt^{-\gamma}$, $0 < \gamma < \frac{1}{2}$ and using (47) we obtain for fixed ζ and $t \to 0$ the relations

$$t^{\frac{1}{2}}\ln t\hat{\Omega}_{31} + t^{\frac{1}{2}}\hat{\Omega}_{30} = t^{\gamma}\ln t\, 2\gamma\zeta s_{31}(\theta) + t^{\gamma}\{\zeta[s_{30}(\theta) - \sin\theta] + 2\zeta\ln\zeta\, s_{31}(\theta)\} + O(t^{1-\gamma}\ln t); \tag{C 6a}$$

$$t\Omega_{20} = -t^{1-\gamma}(1/\zeta)\cos\theta + O(t^{1+\gamma}\ln t). \tag{C 6b}$$

It follows from (C 6) that matching is possible only with (48). The solutions to (47 *a–c*) can be written as

$$\psi_{20} = \sum_{n=0}^{\infty} K_n(\bar{a}r)\,(A_{20}^{(n)} \sin n\theta + B_{20}^{(n)} \cos n\theta) + \phi_{20}(r) \cos\theta, \tag{C 7a}$$

$$\hat{\psi}_{31} = \sum_{n=0}^{\infty} \eta^n (\hat{A}_{31}^{(n)} \sin n\theta + \hat{B}_{31}^{(n)} \cos n\theta), \tag{C 7b}$$

$$\hat{\psi}_{30} = \sum_{n=0}^{\infty} \eta^n (\hat{A}_{30}^{(n)} \sin n\theta + \hat{B}_{30}^{(n)} \cos n\theta) + \varphi_{30}^{(1)}(\eta) \sin\theta + \varphi_{30}^{(2)}(\eta) \cos\theta. \tag{C 7c}$$

Here the functions $\varphi_{30}^{(1)}$, $\varphi_{30}^{(2)}$ are given by (51),

$$\phi_{20}(r) = -\frac{1}{\bar{a}^2 - 1} K_1(r),$$

and $A_{20}^{(n)}$, $B_{20}^{(n)}$, $\hat{A}_{30}^{(n)}$, $\hat{B}_{30}^{(n)}$, $\hat{A}_{31}^{(n)}$, $\hat{B}_{31}^{(n)}$ are constants which must be found by matching the lowest-order sums. The matching procedure is analogous to that for the vorticity expansions (C 5 *a*, *b*) and gives

$$A_{20}^{(n)} = B_{20}^{(n)} = \hat{A}_{30}^{(n)} = \hat{B}_{30}^{(n)} = \hat{A}_{31}^{(n)} = \hat{B}_{31}^{(n)} = 0 \quad \text{for} \quad n \neq 1; \quad A_{20}^{(1)} = 0, \quad A_{31}^{(1)} = 0,$$

$$B_{20}^{(1)} = \frac{\bar{a}}{1-\bar{a}^2}, \quad \hat{A}_{30}^{(1)} = \tfrac{1}{8}\pi, \quad \hat{B}_{31}^{(1)} = -\tfrac{1}{4} \quad \text{for} \quad n = 1,$$

$\hat{B}_{30}^{(1)}$ is given by (50).

Appendix D

We now consider some details of (21), (22) for $\bar{a} = 0$ and $t \to 0$. To show that (46) is inapplicable for large r we write the equation for ψ_{40}:

$$\nabla^2 \psi_{40} = -\frac{1}{2}\left\{\frac{\partial \psi_{20}}{\partial x} + \frac{K_1(r)}{r}\frac{\partial(\nabla^2\psi_{20} - \psi_{20})}{\partial\theta} + [(\tfrac{1}{2}U_{21} - U_{20})\cos\theta + (\tfrac{1}{2}V_{21} - V_{20})\sin\theta]K_1(r)\right\}, \tag{D 1}$$

where U_{21}, U_{20}, V_{21}, V_{20} are the coefficients in expansions (B 7). It follows from (57), (D 1) that

$$\psi_{40} = -\tfrac{1}{8}\cos 2\theta + \bar{s}(r,\theta), \tag{D 2}$$

where $\bar{s}(r,\theta) = o(r^{-1})$ for large r. By virtue of (D 2) the energy of the regular component is infinite for arbitrarily small t.

Substituting (56) into (22) we obtain the following equation for $\bar{\psi}_{20}$:

$$X\frac{\partial \bar{\nabla}^2 \bar{\psi}_{20}}{\partial X} + Y\frac{\partial \bar{\nabla}^2 \bar{\psi}_{20}}{\partial Y} + 4\bar{\nabla}^2\bar{\psi}_{20} + \frac{\partial \bar{\psi}_{20}}{\partial X} = 0, \tag{D 3}$$

where $\bar{\nabla}^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$. The boundary conditions for (D 3) follow from the requirements that $\bar{\psi}_{20}$ must decay as $r \to \infty$ and $t^2\bar{\psi}_{20}$ should be matched with $t^2\psi_{20}$. Let us consider the following auxiliary problem:

$$\frac{\partial \nabla^2 S}{\partial t} + \frac{\partial S}{\partial x} = \frac{\partial K_0}{\partial x}, \tag{D 4a}$$

$$S = 0 \quad \text{for} \quad t = 0, \tag{D 4b}$$

$$S \quad \text{is regular as} \quad r \to 0 \quad \text{and decays as} \quad r \to \infty. \tag{D 4c}$$

Using the smallness of the function ψ_r and its derivatives for $t \to 0$ in regions II and III (see (46), (56)) it can readily be shown that ψ_r approximately satisfies (D 4a) for $t \ll 1$ in regions II, III. We introduce function Π by the formula

$$S = \frac{\partial \Pi}{\partial x}. \tag{D 5}$$

By virtue of (D 4a), Π satisfies the following equation:

$$\frac{\partial \nabla^2 \Pi}{\partial t} + \frac{\partial \Pi}{\partial x} = K_0(r). \tag{D 6}$$

The solution to (D 6) can be written as (Kamenkovich 1989)

$$\Pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0\{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}\}\, G(\xi,\eta,t)\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{D 7}$$

$$G(x,y,t) = -\frac{t}{\pi}\frac{1}{(rt)^{\frac{1}{2}}}\int_0^{\infty} \frac{I_1\{2[rt(u^2+\cos^2(\frac{1}{2}\theta))]^{\frac{1}{2}}\}}{[(u^2+1)(u^2+\cos^2(\frac{1}{2}\theta))]^{\frac{1}{2}}}\mathrm{d}u. \tag{D 8}$$

Here G is the Green's function that is the solution to the problem:

$$\frac{\partial \nabla^2 G}{\partial t} + \frac{\partial G}{\partial x} = H(t)\,\delta(x)\,\delta(y), \tag{D 9a}$$

$$G = 0 \quad \text{for} \quad t = 0, \qquad G < \infty \quad \text{for} \quad r \to \infty, \tag{D 9b}$$

where $H(t)$ is the Heaviside function. We now find the function Π in the case when x, y are of $O(1)$ and $t \to 0$, and in the case when X, Y are of $O(1)$ and $t \to 0$. Here it is advisable to use the fact that

$$\Pi = -2\pi\phi, \tag{D 10}$$

where ϕ is the solution to the equation

$$\nabla^2\phi - \phi = G(x,y,t) \tag{D 11}$$

vanishing at $t = 0$. We have $G \approx (t/2\pi)\ln rt$ for $t \to 0$ and $r = O(1)$ (Kamenkovich 1989), that is for these r and t (D 11) becomes

$$\nabla^2\phi - \phi \approx \frac{1}{2\pi}t\ln t + \frac{1}{2\pi}t\ln r. \tag{D 12}$$

By virtue of (D 12),

$$\phi \approx -\frac{1}{2\pi}t\ln t - \frac{t}{2\pi}[\ln r + K_0(r)],$$

and therefore (D 5), (D 10) yield

$$S = t\hat{S}(x,y) = \frac{\partial \Pi}{\partial x} \approx t\left[\frac{1}{r} - K_1(r)\right]\cos\theta. \tag{D 13}$$

Hence, the function S approximately coincides with $t\psi_{20}$ for $r = O(1)$, $t \to 0$ (see (57)). To find ϕ for $t \to 0$ and fixed X, Y we write (D 11) in terms of X, Y:

$$t^2\bar{\nabla}^2\phi - \phi = t\bar{G}(R,\theta), \tag{D 14}$$

where $R = tr$, $t\bar{G}(R,\theta) = G(x,y,t)$. We seek ϕ in the form of the expansion

$$\phi = t\phi_1(X,Y) + t^2\phi_2(X,Y) + \dots. \tag{D 15}$$

Substituting (D 15) into (D 14) we obtain

$$\phi_1 = -t\bar{G}(R,\theta), \quad \phi_2 = 0, \quad \phi_3 = \nabla^2 \phi_1. \tag{D 16}$$

Accordingly, the solution S has the following form in this region:

$$S = t^2 \bar{S}(X,Y) = -2\pi \frac{\partial \phi}{\partial x} = 2\pi t^2 \frac{\partial \bar{G}}{\partial X} + O(t^3). \tag{D 17}$$

It can easily be shown that in regions II, III the expanisons for S can be matched. In the intermediate region we have

$$\xi = rt^\alpha = O(1), \quad 0 < \alpha < 1, \tag{D 18}$$

and

$$t\hat{S}(x,y) \approx t^{1+\alpha} \frac{1}{\xi} \cos\theta. \tag{D 19}$$

Further we have $\bar{G}(R,\theta) \approx (1/2\pi)\ln R$, $R \rightarrow 0$, i.e. $\partial\bar{G}/\partial X \approx (1/2\pi R)\cos\theta$ as $R \rightarrow 0$ (Kamenkovich 1989). In the intermediate region we have $R = rt = \xi t^{1-\alpha}$, and therefore

$$t^2 \bar{S}(X,Y) = 2\pi t^2 \frac{\partial \bar{G}}{\partial X} \approx t^{1+\alpha} \frac{1}{\xi} \cos\theta. \tag{D 20}$$

Thus, the function $t^2\bar{S} = 2\pi t^2 \partial\bar{G}/\partial X = 2\pi\, \partial G/\partial x$ is matched with $t\psi_{20}$. Finally, by virtue of (D 9), $t^2\bar{S}$ satisfies the equation

$$\frac{\partial \nabla^2 (t^2 \bar{S})}{\partial t} + \frac{\partial (t^2 \bar{S})}{\partial x} = 0, \tag{D 21}$$

whence it readily follows that S satisfies (D 3). Thus the function

$$\bar{\psi}_{20}(X,Y) = 2\pi \frac{\partial \bar{G}}{\partial X}$$

is the desired solution to (D 3).

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